Int. J. Solids Structures, 1973, Vol. 9, pp. 671 to 691, Pergamon Press. Printed in Great Britain

BUCKLING OF UNSTIFFENED AND RING STIFFENED CYLINDRICAL SHELLS UNDER AXIAL COMPRESSION

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Abstract—Bifurcation stresses and initial postbuckling behavior of both unstiffened and outside ring stiffened circular cylindrical shells under axial compression are analyzed. The shells are assumed to have axisymmetric sinusoidal imperfections with arbitrary wavelengths and amplitudes. It is found that for large imperfection amplitudes and wavelengths both the unstiffened and the stiffened shells have extremely small bifurcation loads. The postbuckling analysis shows that for small imperfection amplitudes the bifurcations from the axisymmetric state are initially unstable and collapse is associated with the bifurcation points. However, for larger values of the imperfection amplitudes the bifurcations are stable. For unstiffened shells the transition from unstable to stable bifurcations can take place at very small values of the load. On the other hand, for stiffened shells it is found that bifurcations at load levels less than about 40 per cent of the classical buckling load have stable initial postbuckling behavior.

NOTATION

A	cross sectional area of a ring
B	classical buckling expression (equation (3.2))
b	postbuckling coefficient (equation (4.14))
c	$[3(1-\nu^2)]^{\frac{1}{2}}$
d	spacing of the rings
D	$Et^3/12(1-\nu^2)$
D_{yy}	effective bending stiffness (equation (2.7))
e	eccentricity of the rings
E	Young's modulus
F	stress function
f	$2cF/Et^3$
H_{xx}, H_{xy}, H_{yy}	effective stretching stiffness (equation (2.7))
I	integral (equation (4.17))
k	longitudinal wavenumber
q_0	$[12(1-\nu^2)]^{\frac{1}{4}}[R/t]^{\frac{1}{2}}$
Q_{xy}, Q_{yy}	see equation (2.7)
R	cylinder radius
R_{pq}, S_{pq}	see equation (4.11)
S, S_0	postbuckling and prebuckling axial stiffnesses defined by $\frac{t}{cR}\left(\frac{\partial\lambda}{\partial\Delta}\right)_{\lambda=\lambda_c}$ where Δ is the average shortening per unit length
s	circumferential wavenumber
t	shell thickness
U, V, W	axial, circumferential, and radial displacements
$\tilde{W}$	initial radial displacement (imperfection)
w	W/t
X, Y	longitudinal and circumferential coordinates
x, y	Xq_0/R and Yq_0/R
α	postbuckling parameter (equation (4.18))

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α_r	area parameter for rings (equation (2.2))
β	wavenumber for imperfection
β_r	stiffness parameter for rings (equation (2.2))
γ	$2c\delta/t$
γ_r	eccentricity parameter for rings (equation (2.2))
δ	imperfection amplitude
λ	average axial compression/(Et/cR)
λ_c	value of λ at bifurcation buckling
λ_{cl}	value of λ at classical buckling
ν	Poisson ratio
ξ	amplitude of buckling displacement
τ	$2k/s^2$
Operators	
	$(\dot{\ }) = \dfrac{\partial(\ )}{\partial y}, \quad (\ )' = \dfrac{\partial(\ )}{\partial x}$
	ψ see equation (2.6)
	L_D, L_Q, L_H see equation (2.7)
	$L_p, D_{pq}, Q_{pq}, H_{pq}$ see equation (4.10)

1. INTRODUCTION

THE theoretical significance of allowable buckling stresses used in the design of axially compressed cylindrical shells has been the subject of extensive research. It is now a well established fact that small deviations in the shape of the shell yield drastic reductions in the buckling loads. Koiter [1] has derived a simple formula for the upper bound of the load for which non-axisymmetric bifurcation from the axisymmetric prebuckled state occurs for unstiffened circular cylindrical shells under axial compression. An imperfection in the shape of the classical axisymmetric buckling mode with an amplitude of one shell wall thickness was found to reduce the bifurcation load to about 20 per cent of the classical value. Hutchinson and Amazigo [2] have shown that a similar imperfection in lightly outside ring stiffened shells reduces the bifurcation load to about 45 per cent of the classical buckling load. When amplitudes and wavelengths of the imperfections get larger the bifurcation loads for unstiffened shells, as well as for ring stiffened shells, tend to very small values.

Test results for unstiffened shells under compression [4] give buckling loads for normal shell geometries which are about 30 per cent of the classical value and as low as even 12 per cent for very thin shells. For comparison, the buckling loads for outside ring stiffened cylindrical shells under axial compression have been observed in laboratory tests ([5]–[7]) to vary from 67 to 97 per cent of those given by the classical buckling load.

Recently Budiansky and Hutchinson [3] investigated the stability of bifurcations of axially compressed unstiffened cylindrical shells with axisymmetric initial imperfections in the shape of the classical buckling modes. This investigation shows that when the imperfection amplitude is small, bifurcations from the axisymmetric state are initially unstable and collapse will occur at the bifurcation loads. However, for larger imperfection amplitudes, the bifurcations are stable and thus loads above the bifurcation loads can be sustained. The results indicate that this transition from unstable to stable bifurcation takes place at loads about one third of the classical buckling load.

The work presented here is a similar investigation of bifurcation stresses and post-buckling behavior of axially compressed outside ring stiffened and unstiffened cylindrical shells. The shells are assumed to have axisymmetric sinusoidal imperfections with arbitrary wavelengths and amplitudes. For the unstiffened shell it is shown that for increasing wavelengths of the imperfection, the transition from unstable to stable postbuckling behavior takes place at decreasing bifurcation loads. Thus, the theoretical significance of buckling loads about one third of the classical buckling load indicated in [3] is not observed in the present analysis since smaller transition loads are obtained for longer wavelengths of the initial imperfection.

Compared to the unstiffened shells, the stiffened shells are found to have a much more stable postbuckling behavior. Bifurcations which take place at loads less than about 40 per cent of the classical buckling load are found to be stable for all wavelengths of the initial imperfections, except for extremely large imperfection amplitudes. Furthermore, the range of imperfection wavelengths for which the shells are imperfection sensitive is found to be more restricted for the stiffened shell than for the unstiffened shell. Thus the present analysis may explain the relatively high bifurcation loads obtained in tests of outside ring stiffened shells in contrast to those observed for unstiffened shells.

2. BASIC EQUATIONS

Figure 1 shows an infinitely long shell with thickness t stiffened by closely spaced rings. The middle surface of the shell is assumed to have an initial displacement $\tilde{W}$ in the radial direction from a perfect cylindrical surface of radius R. Points on the cylindrical surface are specified by the axial coordinate X and the circumferential coordinate Y. The initial imperfection of the shell is assumed to be axisymmetric and given by

$$\tilde{W} = -\delta \cos(p_x X/R). \tag{2.1}$$

The ring stiffeners are characterized by the dimensionless parameters

$$\alpha_r = A/\mathrm{d}t, \qquad \beta_r = EI_r/Dd \quad \text{and} \quad \gamma_r = e/t \tag{2.2}$$

where A is the cross-sectional area of a ring
d is the spacing
I_r is the moment of inertia about the neutral axis of the ring
e is the eccentricity of the rings

The torsional stiffness and the bending stiffness of the rings tangential to the shell are not taken into account.

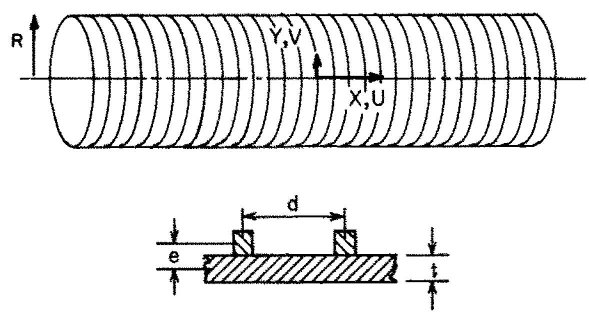

FIG. 1. Part of ring stiffened cylindrical shell.

A set of nonlinear equations for eccentrically stiffened cylindrical shells has been presented by Geier [8] and by Hutchinson and Amazigo [2]. Therefore, in this section we shall just recall the assumptions and present the resulting equations in a non-dimensional form.

According to the Karman–Donnell theory, the middle surface strains in the shell are taken to be

$$\varepsilon_x = \frac{\partial U}{\partial X}+\frac{1}{2}\left(\frac{\partial W}{\partial X}\right)^2, \qquad \varepsilon_y = \frac{\partial V}{\partial Y}+\frac{W}{R}+\frac{1}{2}\left(\frac{\partial W}{\partial Y}\right)^2, \qquad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial U}{\partial Y}+\frac{\partial V}{\partial X}\right)+\frac{1}{2}\frac{\partial W}{\partial X}\frac{\partial W}{\partial Y}, \tag{2.3}$$

where U, V and W are the two tangential and the normal displacements, respectively. The bending strains are given by

$$K_x = -\frac{\partial^2 W}{\partial X^2}, \qquad K_y = -\frac{\partial^2 W}{\partial Y^2} \quad \text{and} \quad K_{xy} = -\frac{\partial^2 W}{\partial X \partial Y}.$$

The bending strain in a ring stiffener is taken to be the same as in the shell, $-\partial^2 W/\partial Y^2$, and the axial strain of the neutral axis is taken to be

$$\varepsilon_r = \varepsilon_y - e\frac{\partial^2 W}{\partial Y^2}.$$

The usual stress–strain relations are assumed for the shell

$$N_x = \frac{Et}{1-\nu^2}(\varepsilon_x+\nu\varepsilon_y) \qquad M_x = D(K_x+\nu K_y)$$

$$N_{xy} = \frac{Et}{1+\nu}\varepsilon_{xy} \qquad M_{xy} = D(1-\nu)K_{xy}$$

$$N_y = \frac{Et}{1-\nu^2}(\varepsilon_y+\nu\varepsilon_x) \qquad M_y = D(K_y+\nu K_x),$$

where D is the plate bending stiffness. For the ring stiffeners we will assume

$$N_r = \frac{EA}{d}\varepsilon_r \quad \text{and} \quad M_r = -\frac{EI}{d}\frac{\partial^2 W}{\partial Y^2}.$$

Using the principle of virtual work under the assumption that the ring stiffeners are closely spaced, three equilibrium equations are obtained. Expressing the total stress resultants by means of an Airy stress function

$$N_x = \frac{\partial^2 F}{\partial Y^2}, \qquad N_y+N_r = \frac{\partial^2 F}{\partial X^2} \quad \text{and} \quad N_{xy} = -\frac{\partial^2 F}{\partial X \partial Y}$$

the three equilibrium equations are reduced to a single equilibrium equation and a compatibility equation. In non-dimensional form the equilibrium equation takes the form

$$L_D[w]+L_Q[f] = 2c\psi(f, w+\tilde{w}) \tag{2.4}$$

and the compatibility equation takes the form

$$L_H[f]-L_Q[w] = -c\psi(w, w+2\tilde{w}), \tag{2.5}$$

where, with $q_0 = [12(1-\nu^2)]^{\frac{1}{4}}[R/t]^{\frac{1}{2}}$, the independent variables are

$$x = Xq_0/R \quad \text{and} \quad y = Yq_0/R$$

and with $c = [3(1-\nu^2)]^{\frac{1}{2}}$, the dependent variables are

$$f = [2c/Et^3]F \quad \text{and} \quad w = W/t.$$

The non-dimensional imperfection is given by $\tilde{w} = -\delta/t \cos \beta x$, where $\beta = p_x/q_0$. When $\beta = 1$, the wavelength of the imperfection corresponds to the wavelength of the classical axisymmetric buckling mode for an unstiffened shell. Values of β less than or greater than one correspond to longer or shorter wavelengths, respectively. The operator ψ is given by

$$\psi(g_1, g_2) = g_1''\ddot{g}_2 + \ddot{g}_1 g_2'' - 2\dot{g}_1'\dot{g}_2' \tag{2.6}$$

where $(\ )' \equiv (\partial/\partial x)(\ )$ and $(\dot{\ }) \equiv (\partial/\partial y)(\ )$. The three linear operators are defined by

$$\begin{aligned} L_D[\ ] &= [\ ]'''' + 2[\ddot{\ }]'' + D_{yy}[\ddot{\ddot{\ }}] \\ L_Q[\ ] &= 2Q_{xy}[\ddot{\ }]'' + Q_{yy}[\ddot{\ddot{\ }}] + [\ ]'' \\ L_H[\ ] &= H_{xx}[\ ]'''' + 2H_{xy}[\ddot{\ }]'' + H_{yy}[\ddot{\ddot{\ }}] \end{aligned} \tag{2.7}$$

where

$$D_{yy} = 1 + \beta_r + \frac{12(1-\nu^2)\alpha_r\gamma_r^2}{1+\alpha_r}$$

$$Q_{xy} = -\frac{c\alpha_r\gamma_r}{1+\alpha_r}, \qquad Q_{yy} = \frac{2c\nu\alpha_r\gamma_r}{1+\alpha_r}$$

$$H_{xx} = \frac{1}{1+\alpha_r}, \qquad H_{xy} = \frac{1+\alpha_r(1+\nu)}{1+\alpha_r}, \qquad H_{yy} = \frac{1+\alpha_r(1-\nu^2)}{1+\alpha_r}.$$

The solutions to be obtained must fulfill the requirement that the tangential displacements are single-valued over any complete circuit of the shell. For the circumferential displacement this condition is enforced if

$$\int_0^{2\pi R} \frac{\partial V}{\partial Y}\, \mathrm{d}Y = 0.$$

Expressed by the non-dimensional stress function f and the normal displacement w this condition takes the form

$$\int_0^{2\pi q_0} \{H_{xx}(f'' - \nu\ddot{f}) - 2Q_{xy}\ddot{w} - w - c(\dot{w})^2\}\, \mathrm{d}y = 0. \tag{2.8}$$

3. AXISYMMETRIC SOLUTION

The nonlinear equations (2.4) and (2.5) admit the following axisymmetric solution for the unbuckled cylinder

$$\begin{aligned} w_0 &= \frac{\nu H_{xx}\lambda}{c} - 2\frac{\delta}{t}\frac{\lambda\beta^2 H_{xx}}{B}\cos\beta x \\ f_0 &= -\frac{\lambda y^2}{2c} + 2\frac{\delta}{t}\frac{\lambda}{B}\cos\beta x \end{aligned} \tag{3.1}$$

which satisfies the condition (2.8). Here

$$B = \beta^4 H_{xx} + 1 - 2\lambda\beta^2 H_{xx} \tag{3.2}$$

and λ is the average axial compressive stress normalized by the classical buckling stress for the unstiffened cylinder (Et/cR). It can be shown that the classical buckling load λ_{cl} for outside ring stiffened cylindrical shells, where the eccentricity of the rings is sufficient to ensure that the corresponding buckling mode is axisymmetric, is given by

$$\lambda_{cl} = H_{xx}^{-\frac{1}{2}}. \tag{3.3}$$

The corresponding wavelength of the axisymmetric buckling mode is determined by the wave parameter

$$\beta_{cl} = H_{xx}^{-\frac{1}{4}}. \tag{3.4}$$

The expressions (3.3) and (3.4) are also valid for unstiffened shells, where $H_{xx} = 1$.

4. BIFURCATION AND POSTBUCKLING BEHAVIOR

The possibilities for bifurcation buckling at load levels less than the classical buckling loads will now be investigated. The method which will be used to analyze the bifurcation stresses and load carrying capacities at the bifurcation points is similar to the method used by Budiansky and Hutchinson [3] for a long unstiffened cylindrical shell containing an axisymmetric imperfection in the shape of the classical buckling mode. This method allows the bifurcation load to be investigated for a complete family of modes and gives an exact formulation of the postbuckling behavior.

Following this procedure it will be assumed that the postbuckling behavior at a load level λ in the vicinity of the bifurcation point λ_c can be written in the form

$$\begin{Bmatrix} w \\ f \end{Bmatrix} = \begin{Bmatrix} w_0(\lambda) \\ f_0(\lambda) \end{Bmatrix} + \xi \begin{Bmatrix} w_1 \\ f_1 \end{Bmatrix} + \xi^2 \begin{Bmatrix} w_2 \\ f_2 \end{Bmatrix} + \dots \tag{4.1}$$

$$\lambda/\lambda_c = 1 + b\xi^2 + \dots \tag{4.2}$$

where (w_1, f_1) is the normalized buckling mode, (w_2, f_2) is a set of functions orthogonal to (w_1, f_1) so that $\int_A w_2 w_1 \, dx\, dy = 0$, ξ is a scalar parameter giving the buckling mode amplitude, and b is a postbuckling coefficient which for the case of a nonlinear prebuckling state has been derived by Fitch [9]. In the present problem it is determined by

$$\frac{b}{1-v^2} = \frac{3}{c\lambda_c} \frac{\langle \ddot{f}_2(w_1')^2 + f_2''(\dot{w}_1)^2 - 2\dot{f}_2'\dot{w}_1 w_1' + 2\ddot{f}_1 w_1' w_2' + 2f_1''\dot{w}_1\dot{w}_2 - 2\dot{f}_1'(\dot{w}_1 w_2' + w_1'\dot{w}_2)\rangle}{\langle (w_1')^2 + \gamma\beta^2[(\beta^4 H_{xx}+1)/B^2][(\dot{w}_1)^2 \cos\beta x - 2\beta H_{xx}(\ddot{f}_1 w_1' - \dot{f}_1'\dot{w}_1)\sin\beta x]\rangle} \tag{4.3}$$

where $\langle\ \rangle$ denotes averaging over the shell. Substituting equation (4.1) into equations (2.4) and (2.5) gives

$$L_D[w_1] + L_Q[f_1] + 2\lambda_c w_1'' + \beta^2\gamma\frac{\cos\beta x}{B}[2\lambda_c \ddot{w}_1 - (\beta^4 H_{xx}+1)\ddot{f}_1] = 0 \tag{4.4a}$$

$$L_H[f_1] - L_Q[w_1] + \beta^2\gamma(\beta^4 H_{xx}+1)\frac{\cos\beta x}{B}\ddot{w}_1 = 0 \tag{4.4b}$$

for determination of the buckling stress λ_c and the buckling mode (w_1, f_1), and the following equations for the second order contributions (w_2, f_2)

$$L_D[w_2]+L_Q[f_2]+2\lambda_c w_2''+\beta^2\gamma\frac{\cos\beta x}{B}[2\lambda_c\ddot{w}_2-(\beta^4 H_{xx}+1)\ddot{f}_2] = 2c\psi(f_1, w_1) \quad (4.5a)$$

$$L_H[f_2]-L_Q[w_2]+\beta^2\gamma(\beta^4 H_{xx}+1)\frac{\cos\beta x}{B}\ddot{w}_2 = -c\psi(w_1, w_1) \quad (4.5b)$$

here $\gamma = 2c\delta/t$.

Now let us seek the general solutions to the eigenvalue problem (4.4) in the Floquet form

$$\begin{aligned} w_1 &= \mathrm{Re}[w_{11}(x)\,e^{ikx/2}]\cos\frac{sy}{2} \\ f_1 &= \mathrm{Re}[f_{11}(x)\,e^{ikx/2}]\cos\frac{sy}{2}. \end{aligned} \quad (4.6)$$

Here (w_{11}, f_{11}) are complex periodic functions in x with period $2\pi/\beta$, and k and s are axial and circumferential wave parameters. To satisfy periodicity the circumferential wave parameters must fulfill the condition that $q_0 s/2$ is an integer; $s = 0$ corresponds to an axisymmetric solution.

The boundary value problems (4.5) have solutions of the form

$$\begin{aligned} w_2 &= 2cs^2\,\mathrm{Re}\{w_{00}(x)+w_{20}(x)\,e^{ikx}+[w_{02}(x)+w_{22}(x)\,e^{ikx}]\cos sy\} \\ f_2 &= 2cs^2\,\mathrm{Re}\{f_{00}(x)+f_{20}(x)\,e^{ikx}+[f_{02}(x)+f_{22}(x)\,e^{ikx}]\cos sy\}. \end{aligned} \quad (4.7)$$

Using the conditions for single valued circumferential displacement (2.8) and the equations (4.4) to (4.5) it is found that the functions (w_{pq}, f_{pq}) are determined by the coupled complex differential equations

$$\begin{aligned} D_{pq}[w_{pq}]+Q_{pq}[f_{pq}]+L_p^2[2\lambda_c w_{pq}+f_{pq}]-q^2\gamma s^2\beta^2\frac{\cos\beta x}{4B}[2\lambda_c w_{pq}-(\beta^4 H_{xx}+1)f_{pq}] &= -\tfrac{1}{16}R_{pq} \\ H_{pq}[f_{pq}]-Q_{pq}[w_{pq}]-L_p^2[w_{pq}]-q^2\gamma s^2\beta^2(\beta^4 H_{xx}+1)\frac{\cos\beta x}{4B}w_{pq} &= \tfrac{1}{32}S_{pq} \end{aligned} \quad (4.8)$$

for $(p, q) = (1, 1), (0, 2)$ and $(2, 2)$, and by the uncoupled differential equations

$$\begin{aligned} H_{xx}L_p^4[w_{po}]+2\lambda_c H_{xx}L_p^2[w_{po}]+w_{po} &= -\tfrac{1}{16}R_{po} \\ H_{xx}L_p^2[f_{po}] &= w_{po}+\tfrac{1}{32}S_{po} \end{aligned} \quad (4.9)$$

for $p = 0, 2$. Here

$$\begin{aligned} L_p[\ ] &= [\ ]'+\frac{ipk}{2}[\ ] \\ D_{pq}[\ ] &= L_p^4[\ ]-\frac{q^2s^2}{2}L_p^2[\ ]+\frac{q^4s^4}{16}D_{yy}[\ ] \\ Q_{pq}[\ ] &= -\frac{q^2s^2}{2}Q_{xy}L_p^2[\ ]+\frac{q^4s^4}{16}Q_{yy}[\ ] \\ H_{pq}[\ ] &= H_{xx}L_p^4[\ ]-\frac{q^2s^2}{2}H_{xy}L_p^2[\ ]+\frac{q^4s^4}{16}H_{yy}[\ ] \end{aligned} \quad (4.10)$$

and

$$
\begin{aligned}
R_{11} &= 0 & S_{11} &= 0 \\
R_{00} &= H_{xx}(w_{11}\bar{f}_{11})'' + \tfrac{1}{2}\bar{w}_{11}w_{11} & S_{00} &= \bar{w}_{11}w_{11} \\
R_{20} &= H_{xx}L_2^2[w_{11}f_{11}] + \tfrac{1}{2}w_{11}^2 & S_{20} &= w_{11}^2 \\
R_{02} &= L_2^2[w_{11}\bar{f}_{11}] - 4\bar{f}'_{11}L_2[w_{11}] & S_{02} &= L_2^2[w_{11}\bar{w}_{11}] - 4\bar{w}'_{11}L_2[w_{11}] \\
R_{22} &= w_{11}f''_{11} + w''_{11}f_{11} - 2w'_{11}f'_{11} & S_{22} &= 2[w_{11}w''_{11} - (w'_{11})^2].
\end{aligned} \tag{4.11}
$$

From the equations (4.8) and (4.9) it is seen that all the complex functions (w_{pq}, f_{pq}) are periodic with period $2\pi/\beta$. The boundary conditions are found from the fact that all essential different solutions can be obtained by assuming Hermitian symmetry about $x = 0$ and $x = \pi/\beta$. Thus

$$
\begin{aligned}
w_{pq}(-x) &= \overline{w_{pq}(x)} & f_{pq}(-x) &= \overline{f_{pq}(x)} \\
w_{pq}(\pi/\beta + x) &= \overline{w_{pq}(\pi/\beta - x)} & f_{pq}(\pi/\beta + x) &= \overline{f_{pq}(\pi/\beta - x)}.
\end{aligned} \tag{4.12}
$$

It can be shown that a complete family of bifurcation modes can be obtained even if we restrict the free parameter k to the interval $0 \le k \le \beta$. The bifurcation modes (w_1, f_1) and the functions (w_2, f_2) as given by equations (4.6) and (4.7) will in general not be periodic in the axial direction. However, when k takes on a value given by $k = \beta l/m$ where l and m are integers, the functions will be periodic with period $4\pi m/\beta$. The case $k = \beta$ corresponds to the shortest possible wavelength in the axial direction. For unstiffened cylindrical shells approximations to the bifurcation loads for $k = \beta$ when $\beta = 1$ have been given by Koiter [1]. Tennyson and Muggeridge [10] have extended Koiter's results and calculated approximate bifurcation loads for unstiffened cylinders for other values of the imperfection wave parameter β. Decreasing values of $k = \beta l/m$ correspond to increasing axial wavelength of the bifurcation mode. Numerical results corresponding to the range $0 < k < \beta$ have been given for unstiffened cylinders with $\beta = 1$ by Budiansky and Hutchinson [3]. The limit $k = 0$ which corresponds to an infinitely long axial wavelength, has no real physical significance. As will be seen later, this limiting case, for which an exact solution can be found, serves to give lower bounds for the bifurcation loads for a wide range of imperfection amplitudes and wavelengths for stiffened as well as for unstiffened shells.

Now, if we abbreviate equations (4.6) and (4.7) by

$$
\begin{Bmatrix} w_1 \\ f_1 \end{Bmatrix} = \cos\frac{sy}{2}\begin{Bmatrix} \hat{w}_1 \\ \hat{f}_1 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} w_2 \\ f_2 \end{Bmatrix} = \begin{Bmatrix} \hat{w}_{2a} \\ \hat{f}_{2a} \end{Bmatrix} + \cos sy \begin{Bmatrix} \hat{w}_{2b} \\ \hat{f}_{2b} \end{Bmatrix} \tag{4.13}
$$

then for the case $k = \beta l/m$, it is found that the postbuckling coefficient b given by (4.3) can be expressed as

$$
\frac{b}{1-\nu^2} = \frac{\dfrac{3s^2}{c\lambda_c}\displaystyle\int_0^{\pi m/\beta}\left[-\frac{\hat{f}_{2b}}{2}(\hat{w}'_1)^2 + \frac{\hat{w}_1^2}{8}(2\hat{f}''_{2a} - \hat{f}''_{2b}) - \frac{\hat{f}'_{2b}}{2}\hat{w}_1\hat{w}'_1 - \frac{\hat{f}_1\hat{w}'_1}{4}(2\hat{w}'_{2a} + \hat{w}'_{2b}) + \frac{\hat{f}''_1}{2}\hat{w}_1\hat{w}_{2b} - \frac{\hat{f}'_1}{4}\{\hat{w}_1(2\hat{w}'_{2a} - \hat{w}'_{2b}) + 2\hat{w}'_1\hat{w}_{2b}\}\right]dx}{\displaystyle\int_0^{\pi m/\beta}[(\hat{w}'_1)^2 + \gamma s^2\beta^2(\beta^4 H_{xx} + 1)\{\hat{w}_1^2\cos\beta x + 2\beta H_{xx}(\hat{f}_1\hat{w}_1)'\sin\beta x\}/(4B^2)]\,dx} \tag{4.14}
$$

Let S_0 and S denote respectively the nondimensional stiffness in the axisymmetric prebuckled state and the initial overall stiffness in the bifurcation state. Then, from (2.3), (4.1) and (4.2) it can be shown that

$$S_0 = \left[H_{yy} + \frac{\gamma^2\beta^4 H_{xx}}{2B^3}(\beta^4 H_{xx}+1)^2\right]^{-1} \tag{4.15}$$

$$S = \left[\frac{1}{S_0} + \frac{3\beta}{4\pi m\lambda_c}\left(\frac{1-\nu^2}{b}\right)I\right]^{-1}, \tag{4.16}$$

where

$$\begin{aligned} I &= 2\int_0^{\pi m/\beta} (\hat{w}_1')^2\,dx && \text{for } k \neq \beta \\ I &= 2\int_0^{\pi m/\beta} (\hat{w}_1')^2\,dx + \frac{4\gamma\beta(\beta^4 H_{xx}+1)}{cB}\int_0^{\pi m/\beta} \hat{w}_{2a}'\sin\beta x\,dx && \text{for } k = \beta. \end{aligned} \tag{4.17}$$

The postbuckling behavior can be measured by the change in overall stiffness following bifurcation. For the results to be presented in Section 8, this change is measured by the nondimensional parameter α defined by

$$\alpha = \frac{2}{\pi}\operatorname{Arctan}\left(\frac{S}{S_0 - S}\right). \tag{4.18}$$

5. NUMERICAL METHOD

For the numerical solution of the eigenvalue problem given by (4.8) for $(p, q) = (1, 1)$ and of the boundary value problems given by equation (4.8) for $(p, q) = (0, 2), (2, 2)$ and by equation (4.9) for $(p, q) = (0, 0)$ and $(2, 0)$, the differential equations are approximated by finite difference expressions in the interval $0 \leq x \leq \pi/\beta$. Introducing N points in the interval for x, spaced at a constant distance h, the finite difference approximation for the differential equations (4.8) at an interior point x_j is taken to be

$$\begin{aligned} &\bar{a}_2(w_{pq})_{j-2} + \left\{\bar{a}_1 + \left(2\lambda_c - \frac{q^2 s^2}{2}\right)\bar{b}_1\right\}(w_{pq})_{j-1} \\ &+ \left\{a_0 + \left(2\lambda_c - \frac{q^2 s^2}{2}\right)b_0 + \frac{q^4 s^4}{16}D_{yy} - \lambda_c q^2\beta^2\gamma s^2\frac{\cos(\beta x_j)}{2B}\right\}(w_{pq})_j \\ &+ \left\{a_1 + \left(2\lambda_c - \frac{q^2 s^2}{2}\right)b_1\right\}(w_{pq})_{j+1} + a_2(w_{pq})_{j+2} + \left\{\left(1 - \frac{q^2 s^2}{2}Q_{xy}\right)\bar{b}_1\right\}(f_{pq})_{j-1} \\ &+ \left\{\left(1 - \frac{q^2 s^2}{2}Q_{xy}\right)b_0 + \frac{q^4 s^4}{16}Q_{yy} + \beta^2 q^2\gamma s^2(\beta^4 H_{xx}+1)\frac{\cos(\beta x_j)}{4B}\right\}(f_{pq})_j \\ &+ \left\{\left(1 - \frac{q^2 s^2}{2}Q_{xy}\right)b_1\right\}(f_{pq})_{j+1} = -\left[\frac{R_{pq}}{16}\right]_{x=x_j} \end{aligned} \tag{5.1}$$

and

$$
\begin{aligned}
&\{H_{xx}\bar{a}_2\}(f_{pq})_{j-2}+\left\{H_{xx}\bar{a}_1-\frac{q^2s^2}{2}H_{xy}\bar{b}_1\right\}(f_{pq})_{j-1}\\
&\quad+\left\{H_{xx}a_0-\frac{q^2s^2}{2}H_{xy}b_0+\frac{q^4s^4}{16}H_{yy}\right\}(f_{pq})_j+\left\{H_{xx}a_1-\frac{q^2s^2}{2}H_{xy}b_1\right\}(f_{pq})_{j+1}\\
&\quad+\{H_{xx}a_2\}(f_{pq})_{j+2}-\left\{\left(1-\frac{q^2s^2}{2}Q_{xy}\right)\bar{b}_1\right\}(w_{pq})_{j-1}-\left\{\left(1-\frac{q^2s^2}{2}Q_{xy}\right)b_0\right.\\
&\quad\left.+\frac{q^4s^4}{16}Q_{yy}+\beta^2q^2\gamma s^2(\beta^4H_{xx}+1)\frac{\cos(\beta x_j)}{4B}\right\}(w_{pq})_j\\
&\quad-\left\{\left(1-\frac{q^2s^2}{2}Q_{xy}\right)b_1\right\}(w_{pq})_{j+1}=\left[\frac{S_{pq}}{32}\right]_{x=x_j}.
\end{aligned}
\tag{5.2}
$$

Here

$$
a_0=\left(\frac{6}{h^4}+\frac{p^4k^4}{16}+3\frac{p^2k^2}{h^2}\right),\qquad a_2=\left(\frac{1}{h^4}+\mathrm{i}\frac{pk}{h^3}\right)
$$

$$
a_1=-\left(\frac{4}{h^4}+\mathrm{i}2\frac{pk}{h^3}+\frac{3p^2k^2}{2h^2}+\mathrm{i}\frac{p^3k^3}{4h}\right)
$$

$$
b_0=-\left(\frac{2}{h^2}+\frac{p^2k^2}{4}\right),\qquad b_1=\frac{1}{h^2}+\mathrm{i}\frac{pk}{2h}.
$$

If the point corresponding to $x = 0$ is denoted by the index 1 and the point corresponding to $x = \pi/\beta$ by the index N, then the boundary conditions take the form

$$
(\bar{w}_{pq})_{-1}=(w_{pq})_3,\qquad (\bar{w}_{pq})_0=(w_{pq})_2
$$

$$
(\bar{w}_{pq})_{N-1}=(w_{pq})_{N+2},\qquad (\bar{w}_{pq})_{N-1}=(w_{pq})_{N+1}.
$$

Similar expressions can be used for f_{pq}.

Due to the form of the boundary conditions, it is advantageous to separate the equations into real and imaginary parts. Thus, the unknowns of the problem are the values of $\mathrm{Re}(w_{pq})$, $\mathrm{Im}(w_{pq})$, $\mathrm{Re}(f_{pq})$, and $\mathrm{Im}(f_{pq})$ at N points.

Bifurcation stresses

For given values of k and of the imperfection parameters γ and β, the bifurcation stress λ_c is determined by a trial and error method as the smallest value which makes the determinant of the coefficients of the homogeneous difference equations corresponding to $p = q = 1$ equal to zero. The circumferential wavenumber s is treated as a free parameter and is chosen so that the eigenvalue λ_c attains a minimum. The determinant is evaluated by means of Gaussian reduction.

For values of k in the range $0 < k < \beta$, the coefficient matrix is a band matrix with a bandwidth 19 and the problem involves $4N$ unknowns. However, in the case $k = \beta$ the numerical work can be reduced considerably by taking the real functions

$$
\zeta(x)=\mathrm{Re}(w_{11}\,e^{i\beta x/2})\quad\text{and}\quad\phi(x)=\mathrm{Re}(f_{11}\,e^{i\beta x/2})
$$

as the unknowns. The corresponding boundary conditions are obtained from

$$\zeta(-x) = \zeta(x), \qquad \zeta\left(\frac{\pi}{\beta}-x\right) = -\zeta\left(\frac{\pi}{\beta}+x\right)$$

and analogous for $\phi(x)$. The coefficient matrix of the difference equations for this problem has a bandwidth 9 and the problem involves only $2N$ unknowns.

When an eigenvalue λ_c is determined with a prescribed accuracy the corresponding eigenfunctions (w_{11}, f_{11}) are determined by back substitution and normalized.

When the imperfection amplitude γ tends to infinity the circumferential wave parameter s is found to approach zero. Therefore, limiting results for the bifurcation stress and the postbuckling behavior for $\gamma \to \infty$ are found by letting $s = 0$ in the difference equations when s is not multiplied by γ and then introducing a new parameter ρ for γs^2. Instead of minimizing the bifurcation load λ_c with respect to s, λ_c is in this case minimized with respect to the parameter ρ.

Postbuckling behavior

When the bifurcation stress λ_c and the corresponding eigenfunctions (w_1, f_1) are determined, the right hand sides of the differential equations (5.1) and (5.2) for $(p, q) = (0, 2)$ and $(2, 2)$ can be determined by numerical differentiation. The resulting sets of non-homogeneous linear equations are solved by means of Gaussian elimination. The functions (w_{00}, f_{00}) and (w_{20}, f_{20}) are determined in the same way by a simpler set of difference equations corresponding to the differential equations (4.9).

The postbuckling coefficient b and the integral I are evaluated by numerical integration using Simpson's method.

6. ASYMPTOTIC ANALYSIS FOR $k \to 0$

In [3] it was shown that for an unstiffened shell with imperfections in the shape of the classical buckling mode, a quadratic relation exists between the circumferential wave parameter s and k, when k tends to zero. It is reasonable to assume that a similar relation exists for stiffened shells and for other wavelengths of the imperfection. Thus, in the equations (4.8) and (4.9) we will in the limit substitute k by $\tau s^2/2$, where τ is a free parameter the value of which will be determined so that the bifurcation load attains a minimum. Then the solutions to the equations (4.8) and (4.9) in the vicinity of the bifurcation point can be written as the expansion

$$\begin{Bmatrix} w_{pq} \\ f_{pq} \end{Bmatrix} = \begin{Bmatrix} \overset{0}{w}_{pq} \\ \overset{0}{f}_{pq} \end{Bmatrix} + s^2 \begin{Bmatrix} \overset{1}{w}_{pq} \\ \overset{1}{f}_{pq} \end{Bmatrix} + s^4 \begin{Bmatrix} \overset{2}{w}_{pq} \\ \overset{2}{f}_{pq} \end{Bmatrix} + \dots \tag{6.1}$$

$$\lambda_c = \lambda_0 + s^2 \lambda_1 + \dots$$

The eigenfunctions $(\overset{0}{w}_{11}, \overset{0}{f}_{11})$ are assumed normalized in some way and the expansions $(\overset{l}{w}_{11}, \overset{l}{f}_{11})$ $(l > 0)$ are determined so that they are orthogonal to $(\overset{0}{w}_{11}, \overset{0}{f}_{11})$:

$$\int_0^{2\pi/\beta} \overset{l}{w}_{11} \overset{0}{w}_{11} \, dx = 0. \tag{6.2}$$

Solutions for the cases $(p, q) = (1, 1)$, $(0, 2)$ and $(2, 2)$ are obtained by substituting the expansion (6.1) into (4.8). The expansions of the right hand sides are denoted by $\overset{0}{R}_{pq}, \overset{1}{R}_{pq}, \ldots$ and $\overset{0}{S}_{pq}, \overset{1}{S}_{pq}, \ldots$. Then the equations for $(\overset{0}{w}_{pq}, \overset{0}{f}_{pq})$ take the form

$$\begin{aligned} \overset{0}{w}{}''''_{pq} + 2\lambda_0 \overset{0}{w}{}''_{pq} + \overset{0}{f}{}''_{pq} &= -\tfrac{1}{16}\overset{0}{R}_{pq} \\ H_{xx}\overset{0}{f}{}''''_{pq} - \overset{0}{w}{}''_{pq} &= \tfrac{1}{32}\overset{0}{S}_{pq}. \end{aligned} \tag{6.3}$$

These equations have solutions of the form $(\overset{0}{w}_{pq}, \overset{0}{f}_{pq}) = (a_p, c_p)$ where a_p, c_p are constants. If we use this solution, the equations for $(\overset{1}{w}_{pq}, \overset{1}{f}_{pq})$ and $(\overset{2}{w}_{pq}, \overset{2}{f}_{pq})$ take the form

$$\begin{aligned} \overset{1}{w}{}''''_{pq} + 2\lambda_0 \overset{1}{w}{}''_{pq} + \overset{1}{f}{}''_{pq} &= q^2\gamma\beta^2 \frac{\cos\beta x}{4B}[2\lambda_0 a_p - (\beta^4 H_{xx} + 1)c_p] - \tfrac{1}{16}\overset{1}{R}_{pq} \\ H_{xx}\overset{1}{f}{}''''_{pq} - \overset{1}{w}{}''_{pq} &= q^2\gamma\beta^2(\beta^4 H_{xx} + 1)\frac{\cos\beta x}{4B}a_p + \tfrac{1}{32}\overset{1}{S}_{pq} \end{aligned} \tag{6.4}$$

and

$$\begin{aligned} \overset{2}{w}{}''''_{pq} + 2\lambda_0 \overset{2}{w}{}''_{pq} + \overset{2}{f}{}''_{pq} = {}& \frac{q^2}{2}\overset{1}{w}{}''_{pq} - \frac{q^4}{16}D_{yy}a_p - \mathrm{i}p\tau \overset{1}{w}{}'''_{pq} + \frac{q^2}{2}Q_{xy}\overset{1}{f}{}''_{pq} - \frac{q^4}{16}Q_{yy}c_p \\ & - 2\lambda_1 \overset{1}{w}{}''_{pq} - \mathrm{i}p\tau\lambda_0 \overset{1}{w}{}'_{pq} + \frac{p^2\tau^2}{8}\lambda_0 a_p - \mathrm{i}\frac{\tau p}{2}\overset{1}{f}{}'_{pq} + \frac{p^2\tau^2}{16}c_p \\ & + q^2\gamma\beta^2\frac{\cos\beta x}{4B}[2\lambda_0 \overset{1}{w}_{pq} + 2\lambda_1 a_p - (\beta^4 H_{xx} + 1)\overset{1}{f}_{pq}] - \tfrac{1}{16}\overset{2}{R}_{pq} \\ H_{xx}\overset{2}{f}{}''''_{pq} - \overset{2}{w}{}''_{pq} = {}& \frac{q^2}{2}H_{xy}\overset{1}{f}{}''_{pq} - \frac{q^4}{16}H_{yy}c_p - \mathrm{i}p\tau H_{xx}\overset{1}{f}{}'''_{pq} - \frac{q^2}{2}Q_{xy}\overset{1}{w}{}''_{pq} \\ & + \frac{q^4}{16}Q_{yy}a_p + \mathrm{i}\frac{\tau}{2}p\overset{1}{w}{}'_{pq} - \frac{p^2\tau^2}{16}a_p + q^2\gamma\beta^2(\beta^4 H_{xx} + 1)\frac{\cos\beta x}{4B}\overset{1}{w}_{pq} + \tfrac{1}{32}\overset{2}{S}_{pq}. \end{aligned} \tag{6.5}$$

The equations (6.4) have solutions of the form

$$\begin{Bmatrix} \overset{1}{w}_{pq} \\ \overset{1}{f}_{pq} \end{Bmatrix} = \cos\beta x \begin{Bmatrix} b_p \\ d_p \end{Bmatrix} + \begin{Bmatrix} g_p \\ h_p \end{Bmatrix} \tag{6.6}$$

where b_p, d_p, g_p and h_p are constants.† If we substitute equation (6.6) in equations (6.4) and (6.5) and invoke periodicity in equation (6.5) we obtain the following set of equations to be solved for (a_p, c_p, b_p, d_p) for $(p, q) = (1, 1), (0, 2)$ and $(2, 2)$

$$M_{pq}\begin{Bmatrix} a_p \\ c_p \\ b_p \\ d_p \end{Bmatrix} = \frac{q-1}{16}\left(8\begin{Bmatrix} 4\beta^2 b_1 d_1 \\ 2\beta^2 b_1^2 \\ c_1 b_1 + a_1 d_1 \\ a_1 b_1 \end{Bmatrix} + (2-p)\tau^2\begin{Bmatrix} 2a_1 c_1 \\ a_1^2 \\ 0 \\ 0 \end{Bmatrix}\right), \tag{6.7}$$

† The orthogonality condition (6.2) is fulfilled for $g_1 = 0$. However, the constants g_p, h_p are not necessary for the evaluation of the buckling load and the postbuckling behavior.

where

$$M_{pq} = \begin{bmatrix} (q^4D_{yy}-2\lambda_0p^2\tau^2) & (q^4Q_{yy}-p^2\tau^2) & -4\lambda_0\gamma\beta^2q^2/B & 2\beta^2\gamma(\beta^4H_{xx}+1)q^2/B \\ (q^4Q_{yy}-p^2\tau^2) & -q^4H_{yy} & 2\beta^2\gamma(\beta^4H_{xx}+1)q^2/B & 0 \\ -4\gamma\lambda_0q^2/B & 2\gamma q^2(\beta^4H_{xx}+1)/B & 8(\beta^2-2\lambda_0) & -8 \\ 2\gamma q^2(\beta^4H_{xx}+1)/B & 0 & -8 & -8H_{xx}\beta^2 \end{bmatrix}.$$

When we substitute (6.1) into (4.9) the perturbation solutions for w_{00} and w_{20} are obtained as

$$w_{p0} = -\tfrac{1}{32}a_1^2+s^2\frac{H_{xx}\beta^2(b_1c_1+a_1d_1)-a_1b_1}{16B}\cos\beta x$$
$$-s^4\left[\frac{b_1^2}{64}+\frac{p^2\tau^2H_{xx}}{256}a_1(\lambda_0a_1-c_1)+\text{trigonometric function}\right] \quad (p=0,2). \quad (6.9)$$

Buckling

In order to have nontrivial solutions of the homogeneous equations given by (6.7) for $(p, q) = (1, 1)$ the determinant of M_{11} must equal zero. This gives an exact characteristic equation for λ_0. Minimization of λ_0 with respect to τ^2 gives

$$\tau^2 = \lambda_0H_{yy}+Q_{yy}+\frac{\gamma^2\beta^2}{2B^3}(\beta^4H_{xx}+1)\{\beta^4H_{xx}+1+2\lambda_0\beta^2H_{xx}+\beta^2\lambda_0H_{xx}(\beta^4H_{xx}+1)\}. \quad (6.10)$$

In the case of a perfect shell, i.e. $\gamma = 0$, the bifurcation load is obtained from the characteristic equation

$$\{H_{yy}(D_{yy}-2\lambda_0\tau^2)+(Q_{yy}-\tau^2)^2\}\{1+\beta^4H_{xx}-2\beta^2\lambda_0H_{xx}\} = 0. \quad (6.11)$$

If the shell under consideration is without stiffening it is seen that for $\beta = 1$ and $\tau = 1$ then $\lambda_0 = 1$ is a double root. For the case of an outside-stiffened cylinder with sufficient eccentricity the second factor in (6.11) equated to zero gives the bifurcation load. The resulting expression is the same as the expression which determines the classical buckling load.

It can be shown that the bifurcation loads even for stiffened shells can take arbitrarily small values when the imperfection wavelength and amplitude become large. For example, if $\gamma \to \infty$ and $\beta \to 0$ so that $\gamma^2\beta^2 = C/2$ where C is a constant, it is found that

$$\tau^2 = C+\lambda_0H_{yy}+Q_{yy}$$

and

$$\lambda_0 = \{-(C+Q_{yy})+[(C+Q_{yy})^2+H_{yy}D_{yy}]^{\frac{1}{2}}\}/H_{yy}.$$

It is easily seen that if the value of C increases, the values of λ_0 decrease and the bifurcation load can take arbitrarily small positive values.

Postbuckling behavior

When the eigenvector (a_1, c_1, b_1, d_1) corresponding to a bifurcation load λ_0 is determined, then (a_p, c_p, b_p, d_p) for $p = 0, 2$ can easily be found from (6.7). Straightforward but fairly lengthy calculations show that in terms of (a_p, c_p, b_p, d_p), the postbuckling coefficient b takes the form

$$\frac{b}{s^4(1-\nu^2)} = \frac{6}{\lambda_0}\Bigg[-\frac{3}{8B}\{\beta^2(b_1c_1+a_1d_1)-a_1b_1\}\{\beta^2H_{xx}(b_1c_1+a_1d_1)-a_1b_1\}$$

$$+\frac{\tau^2 a_1}{32}\{-16c_0a_1-\lambda_0a_1^3H_{xx}-32a_0c_1+(1+H_{xx})a_1^2c_1\}+\tfrac{3}{8}a_1^2b_1^2$$

$$-2\beta^2(c_1b_1+a_1d_1)(b_0+\tfrac{1}{2}b_2)-\beta^2a_1b_1(2d_0+d_2)-2\beta^2b_1^2(2c_0+c_2)$$

$$-4\beta^2b_1d_1(2a_0+a_2)\Bigg]\Bigg/\Bigg[8b_1^2\beta^2+\tau^2a_1^2$$

$$+\frac{4\gamma\beta^2}{B^2}(\beta^4H_{xx}+1)\{a_1b_1-\beta^2H_{xx}(a_1d_1+c_1b_1)\}\Bigg]. \tag{6.12}$$

Having calculated the integral I, which can be expressed as

$$\frac{I}{s^4} = \frac{\pi}{16\beta}(8b_1^2\beta^2+\tau^2a_1^2) \tag{6.13}$$

the stiffnesses S_0 and S and thereafter the postbuckling parameter α are determined using (4.15), (4.16) and (4.17).

7. APPROXIMATE SOLUTION FOR $k = \beta$

By means of the numerical method presented in Section 5 and the exact limiting results for $k = 0$ presented in Section 6 it is found, that over essentially the whole region of imperfection amplitudes and wavelengths studied, the bifurcation loads are bounded by the bifurcation loads corresponding to $k = \beta$ on the one hand and the limiting results for $k = 0$ on the other hand. With simple asymptotical results for bifurcation loads and postbuckling behavior for $k = 0$ it is desirable to have similar simple results for $k = \beta$. Such approximate expressions which are relatively simple will be derived in this section by an extension of the Galerkin method used by Koiter in [1].

Buckling load

When the normal displacement in the buckling mode is approximated by

$$w_1(x, y) = \cos\frac{\beta x}{2}\cos\frac{sy}{2} \tag{7.1}$$

then the compatibility equation (4.4b) is fulfilled if

$$f_1(x, y) = \left(b_1\cos\frac{\beta x}{2}+b_2\cos\frac{3\beta x}{2}\right)\cos\frac{sy}{2}, \tag{7.2}$$

where

$$b_1 = \frac{2Q_{xy}s^2\beta^2 + Q_{yy}s^4 - 4\beta^2 + 2\gamma s^2\beta^2(\beta^4 H_{xx}+1)/B}{\beta^4 H_{xx} + 2s^2\beta^2 H_{xy} + s^4 H_{yy}}$$

$$b_2 = \frac{2\gamma s^2\beta^2(\beta^4 H_{xx}+1)}{B\{81\beta^4 H_{xx} + 18s^2\beta^2 H_{xy} + s^4 H_{yy}\}}.$$

If we insert equations (7.1) and (7.2) into equation (4.4a), multiply by cos $\beta x/2$ and integrate, we find that the approximation to the buckling load λ_c is given by the following characteristic equation

$$\begin{aligned} &B(\beta^4 + 2s^2\beta^2 + s^4 D_{yy} - 8\lambda_c\beta^2) - 4\gamma s^2\beta^2\lambda_c + b_1 B(2s^2\beta^2 Q_{xy} + s^4 Q_{yy} - 4\beta^2) \\ &\quad + 2\gamma s^2\beta^2(\beta^4 H_{xx}+1)(b_1+b_2) = 0. \end{aligned} \tag{7.3}$$

The circumferential wavenumber s is treated as a free parameter and chosen so that the smallest eigenvalue λ_c attains a minimum. The bifurcation buckling stress λ_c obtained from (7.3) will always be an upper bound for the exact value.

In the special case where $\gamma \to \infty$, we can introduce the parameter $\rho = \gamma s^2$. Then the characteristic equation takes the form

$$B^2\{\beta^4 H_{xx} - 8\lambda_c\beta^2 H_{xx} + 16(1-\rho)\} - 36\rho\beta^2\lambda_c B H_{xx} + \tfrac{328}{81}\rho^2(\beta^4 H_{xx}+1)^2 = 0$$

where

$$\rho = \frac{81B(4B + 9s^2\lambda_c H_{xx})}{164(\beta^4 H_{xx}+1)^2}.$$

Postbuckling behavior

Using the approximate bifurcation stress and bifurcation mode we will now determine approximations to (w_2, f_2). Due to the form of the right hand sides of (4.5) we will seek approximations to (w_2, f_2) in the form (4.13).

From (2.8) it is found that single valued circumferential displacements are obtained if

$$H_{xx}\hat{f}''_{2a} = \hat{w}_{2a} + \frac{cs^2}{8}\cos^2\frac{\beta x}{2}. \tag{7.4}$$

Inserting equations (7.1), (7.2) and (7.4) into equation (4.5a) gives

$$\begin{aligned} &H_{xx}\hat{w}''''_{2a} + 2\lambda_c H_{xx}\hat{w}''_{2a} + \hat{w}_{2a} \\ &\quad = -\frac{s^2c}{16}\{1 + (1 - 2\beta^2(b_1+b_2)H_{xx})\cos\beta x - 8\beta^2 H_{xx} b_2 \cos 2\beta x\}. \end{aligned} \tag{7.5}$$

A partial solution to equation (7.5) is

$$\hat{w}_{2a} = \frac{s^2c}{16}\left\{-1 + \frac{2\beta^2 H_{xx}(b_1+b_2) - 1}{B}\cos\beta x + \frac{8\beta^2 H_{xx} b_2}{16H_{xx}\beta^4 - 8\lambda_c H_{xx}\beta^2 + 1}\cos 2\beta x\right\}. \tag{7.6}$$

Approximations to the function $(\hat{w}_{2b}, \hat{f}_{2b})$ will be sought of the form

$$\begin{aligned} \hat{w}_{2b} &= cs^2\left\{v_0 + \left(\frac{B}{8s^2\gamma(\beta^4 H_{xx}+1)} + s^4 v_1\right)\cos\beta x\right\} \\ \hat{f}_{2b} &= cs^2\{g_0 + g_1\cos\beta x + g_2\cos 2\beta x\}. \end{aligned} \tag{7.7}$$

Substituting equation (7.7) into equation (4.5b) gives

$$g_0 = \frac{Q_{yy}}{H_{yy}}v_0 + \frac{s^2\gamma\beta^2(\beta^4H_{xx}+1)}{2H_{yy}B}v_1$$

$$g_1 = \frac{8s^4\gamma^2\beta^2(\beta^4H_{xx}+1)^2v_0 + B(2s^2\beta^2Q_{xy}+s^4Q_{yy}-\beta^2)(B+8s^6\gamma(\beta^4H_{xx}+1)v_1)}{8s^2\gamma B(\beta^4H_{xx}+1)\{\beta^4H_{xx}+2s^2\beta^2H_{xy}+s^4H_{yy}\}}$$

$$g_2 = \frac{s^2\beta^2\gamma(\beta^4H_{xx}+1)}{2B\{16\beta^4H_{xx}+8s^2\beta^2H_{xy}+s^4H_{yy}\}}\left\{\frac{B}{8s^2\gamma(\beta^4H_{xx}+1)}+s^4v_1\right\}.$$

Substitution into equation (4.5a) and application of the Galerkin method gives the following expressions for v_0 and v_1:

$$v_0 = \frac{r_1a_{22}-r_2a_{12}}{a_{11}a_{22}-a_{12}a_{21}} \quad \text{and} \quad v_1 = \frac{r_2a_{11}-r_1a_{21}}{a_{11}a_{22}-a_{12}a_{21}}$$

where

$$a_{11} = s^4D_{yy}+s^4\frac{Q_{yy}^2}{H_{yy}}+\frac{\beta^4s^4\gamma^2(\beta^4H_{xx}+1)^2}{2B^2(\beta^4H_{xx}+2s^2\beta^2H_{xy}+s^4H_{yy})}$$

$$a_{12} = a_{21} = \frac{\beta^2s^2\gamma(\beta^4H_{xx}+1)}{2B}\left[\frac{Q_{yy}}{H_{yy}}+\frac{2Q_{xy}s^2\beta^2+s^4Q_{yy}-\beta^2}{\beta^4H_{xx}+2s^2\beta^2H_{xy}+s^4H_{yy}}\right]-\frac{\beta^2\lambda_c s^2\gamma}{B}$$

$$a_{22} = \frac{s^4}{2}\left[(\beta^4+2\beta^2s^2+s^4D_{yy})+\frac{(2\beta^2s^2Q_{xy}+s^4Q_{yy}-\beta^2)^2}{\beta^4H_{xx}+2s^2\beta^2H_{xy}+s^4H_{yy}}-2\lambda_c\beta^2\right]$$
$$+\frac{\beta^2\gamma^2s^4(\beta^4H_{xx}+1)^2}{4B^2}\left\{\frac{\beta^2}{H_{yy}}+\frac{s^4}{2(16\beta^4H_{xx}+8s^2\beta^2H_{xy}+s^4H_{yy})}\right\}$$

$$r_1 = \frac{\beta^2}{8}b_1+\frac{\beta^2\lambda_c}{8(\beta^4H_{xx}+1)}-\frac{\beta^2(2s^2\beta^2Q_{xy}+s^4Q_{yy}-\beta^2)}{16(\beta^4H_{xx}+2s^2\beta^2H_{xy}+s^4H_{yy})}$$

$$r_2 = \frac{\beta^2}{4}b_2-\frac{Ba_{22}}{8s^6\gamma(\beta^4H_{xx}+1)}+\frac{\beta^4\gamma(\beta^4H_{xx}+1)}{32s^2BH_{yy}}.$$

When the approximations to $(\hat{w}_{2a}, \hat{f}''_{2a})$ and $(\hat{w}_{2b}, \hat{f}_{2b})$ have been obtained, the approximation to the postbuckling coefficient b can be determined from equation (4.14). Note that all the expressions necessary for evaluation of the postbuckling behavior are formulated so that, as discussed before, it is possible to introduce the parameter $\rho = \gamma s^2$ in order to get limiting results for $\gamma \to \infty$.

8. NUMERICAL RESULTS AND DISCUSSION

Figures 2 and 3 show the bifurcation stress λ_c and a measure of the postbuckling stiffness α for unstiffened shells. The bifurcation stress λ_c and the postbuckling stiffness α are shown as functions of the nondimensional imperfection amplitude $\sqrt{(1-\nu^2)}\delta/t$ for various values of the imperfection wavenumber β. Note, that $\beta = 1$ corresponds to the wavelength of the classical buckling mode, and values of β less than or greater than one correspond to longer or shorter wavelengths of the imperfection, respectively. Only results corresponding to $k = \beta$ and $k = 0$, i.e. short axial wavelengths and infinitely long axial wavelengths of the bifurcation mode, are shown. Bifurcation stresses and post-buckling behavior corresponding to k equal to $\beta/4$, $\beta/2$ and $3\beta/4$ were computed for certain values of the parameters involved. In all the cases studied it was found that the bifurcation stresses were essentially bounded by those obtained from $k = 0$ and $k = \beta$. However, it should be emphasized, that the results associated with $k = 0$ have no physical

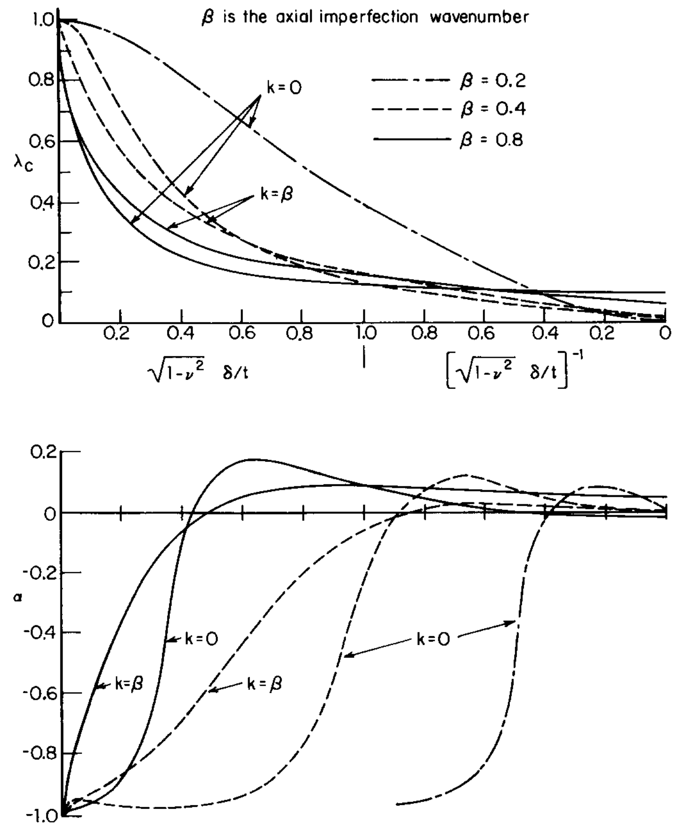

FIG. 2. Buckling and postbuckling of unstiffened cylindrical shell.

significance due to the corresponding infinitely long axial wavelength of the bifurcation mode. The reason for including both the values 0 and β for the bifurcation mode parameter k is to indicate the influence of various axial wavelengths of the bifurcation mode on the bifurcation stress λ_c and the post-buckling stiffness parameter α. The results corresponding to $\beta = 1$ are the same as the results shown in [3].

It is seen that for a fixed imperfection amplitude a critical wavenumber β exists, less than or equal to one, which gives the lowest bifurcation load. Only in the case of a vanishing imperfection amplitude is this value of β equal to one. When the amplitude of the imperfection is large, the critical value of β is small, and as shown in Section 6, the bifurcation

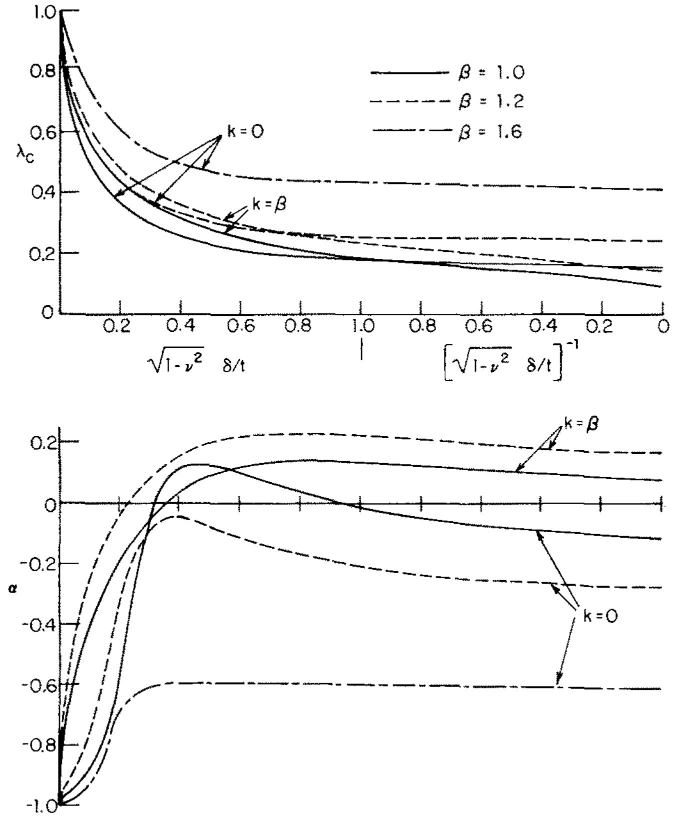

FIG. 3. Buckling and postbuckling of unstiffened cylindrical shell.

loads can take arbitrarily small positive values. For values of the imperfection wavenumber β greater than one, the bifurcation stresses are relatively high even for large imperfection amplitudes.

With respect to the initial postbuckling behavior it is seen that the bifurcations are all unstable for small values of the imperfection amplitude. Except for certain cases corresponding to $k = 0$ and wavenumber β greater than one (where we have relatively high bifurcation stresses), it is seen that for increasing values of the imperfection amplitude a change from unstable to stable bifurcations takes place. In the study by Budiansky and Hutchinson [3], where the imperfection shape is restricted to the classical axisymmetric buckling mode (corresponds to $\beta = 1$ in Fig. 3), the transition from unstable to stable bifurcation is found to take place at loads about one third of the classical buckling load. Thus, the results observed in [3] gave an indication of a theoretical significance of buckling loads about one third of the classical buckling load. However, here we see that for longer wavelengths of the initial imperfection this transition takes place for much smaller values of the bifurcation load. For example, for $\beta = 0{\cdot}2$ and $k = 0$ the transition takes place at bifurcation stresses as low as about one eighth of the classical buckling stress. The corresponding imperfection amplitude is about twice the shell thickness. Therefore, we must conclude that the present analysis does not give theoretical significance to buckling loads for unstiffened shells about one third of the classical buckling load as long as amplitudes and wavelengths of the initial imperfection are not restricted.

Figures 4 and 5 show the results for a lightly ring stiffened circular cylindrical shell with outside rings. The following typical properties of the rings were chosen:

$$\alpha_r = A/dt = 0{\cdot}5, \qquad \beta_r = EI_r/Dd = 20{\cdot}0 \quad \text{and} \quad \gamma_r = e/t = 3{\cdot}0.$$

The classical buckling load of this shell is $\lambda_{cl} = 1{\cdot}2247$ corresponding to the wave parameter $\beta_{cl} = 1{\cdot}107$.

By comparison of the bifurcation stresses for the stiffened shell with those for the unstiffened shell it is found that the range of imperfection wavelengths for which the bifurcation loads are sensitive to small imperfection amplitudes is more restricted for the stiffened shell than for the unstiffened shell. Furthermore, if we consider the initial postbuckling behavior, it is seen that the stiffened shell has a much more stable postbuckling behavior. For example, except for extremely large imperfection amplitudes, the lowest calculated unstable bifurcation takes place at a load about 35 per cent of the classical buckling load. If we only consider those values of k which for fixed imperfection amplitudes and wavenumbers give the smallest value of the bifurcation stress, then the lowest unstable bifurcation stress is found to be 46 per cent of the classical buckling stress. This value is obtained when the wavenumber β equals 0·6 times the wavenumber of the classical buckling mode and the imperfection amplitude is given by $\sqrt{(1-\nu^2)}\delta/t = 1{\cdot}3$.

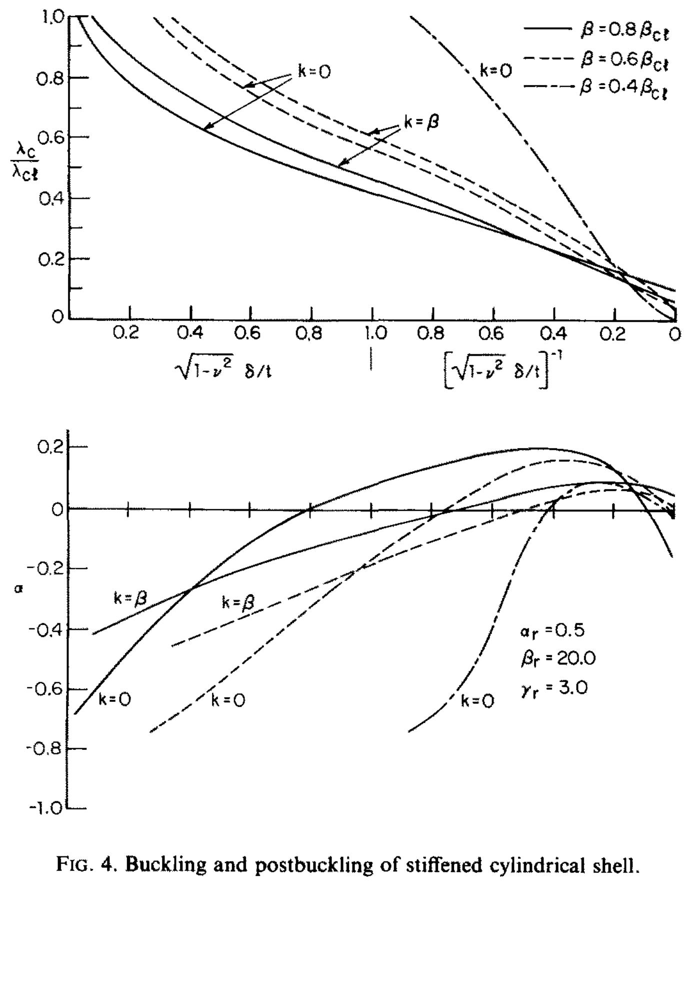

FIG. 4. Buckling and postbuckling of stiffened cylindrical shell.

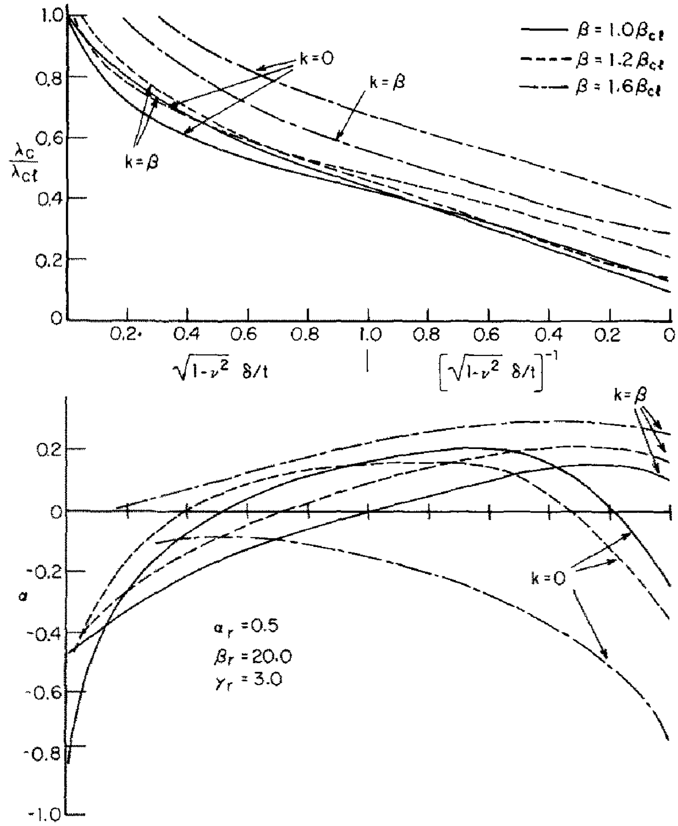

FIG. 5. Buckling and postbuckling of stiffened cylindrical shell.

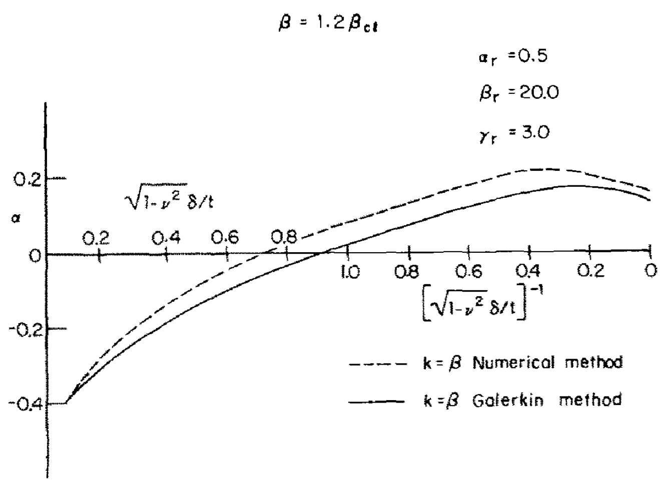

FIG. 6. Postbuckling behavior predicted by the Galerkin method.

It should be emphasized that in the present study we have only determined the initial postbuckling behavior. Of course, even if the initial postbuckling behavior indicates an unstable bifurcation, the possibility exists that the general postbuckling behavior is such that a higher load than the bifurcation load can be supported before total collapse occurs. Also, even if the initial postbuckling behavior is stable it is possible that only a slightly higher load than the bifurcation load can be supported.

The initial postbuckling behavior obtained by the approximate method described in Section 7 is in Fig. 6 compared with the results obtained by the numerical method described in Section 5. The bifurcation stresses obtained by the Galerkin method are not shown since the difference between these results and the bifurcation stresses determined by the numerical procedure for this example does not exceed 1·8 per cent.

Acknowledgements—The author wishes to thank B. Budiansky and J. W. Hutchinson for their advice and suggestions. The work was supported in part by the National Aeronautics and Space Administration under Grant NGL 22–007–012, and by the Division of Engineering and Applied Physics, Harvard University, and this aid is gratefully acknowledged.

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(*Received* 31 *July* 1972; *revised* 13 *November* 1972)

Абстракт—Дается анализ напряжений для бифуркации и начальное поведение в послекритической области, как лдя неподкрепленных так и подкрепленных внешными кольцами, круглых цилиндрических оболочек, под влиянием осебого сжатия. Предполагается, что оболочки имеют осесимметрические, синусоидадьные неточности, с произвольными длинами волн и амплитудами. Находится, что неточностей большой амплитуды и длин волны, как неподкрепленные так и подкрепленные оболочки имеют чрезвичайно малые нагрузки бифуркации. Анализ в послекритической области указывает, что для малых амплитуд неточностей бифуркации от осесимметрического состояния с начала неустойчивы и разрушение находится во взаимно однозначном соответствии с точками бифуркации. Тем не менее, бифуркации устойчивы для больших значений амплитуд неточностей. Для неподкрепленных оболочек, перескок от неустойчивых к устойчивым бифуркациям может происходить для очень малых величин нагрузки. С другой стороны, находится для подкрепленных оболочек, что для уровня нагрузки меньше чем приблизительно 40 процентов классической нагрузки выпучивания, бифуркации имеют устойчивое начальное поведение в послекритической области.